Meshes

Lecture 20

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Outline

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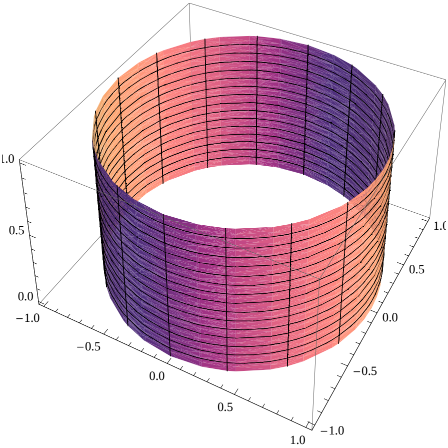
Simple 2D and 3D Surfaces

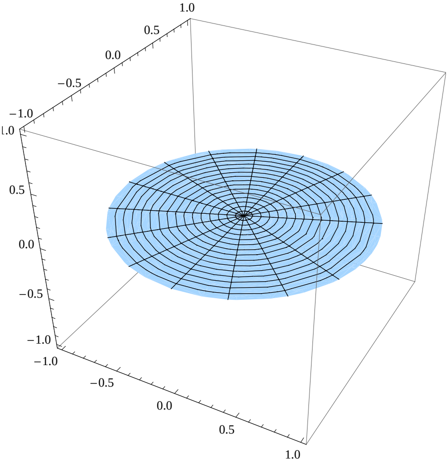
- Whenever possible, we will build complex surfaces from simple surfaces.
- The standard simple surfaces are
  - Square
  - Cylinder
  - Disk
  - Cone
  - Paraboloid
  - Sphere
  - Hyperboloids (or one or two sheets)
  - Torus

Simple 3D Surfaces

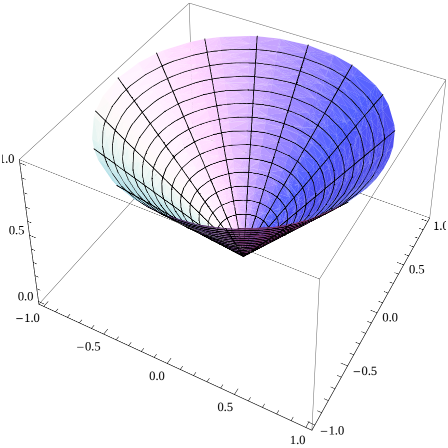


Cylinder: $x^2 + z^2 = 1$, $0 \le y \le 1$

Simple 3D Surfaces

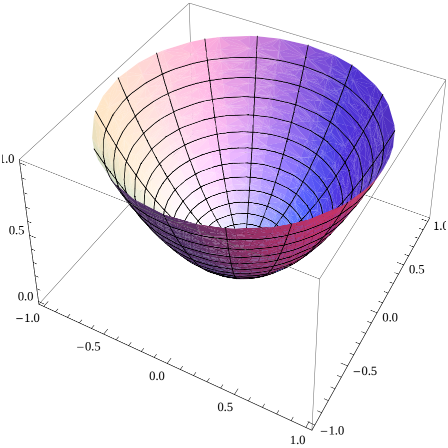


Disk: $x^2 + z^2 \leq 1$, $y = 0$



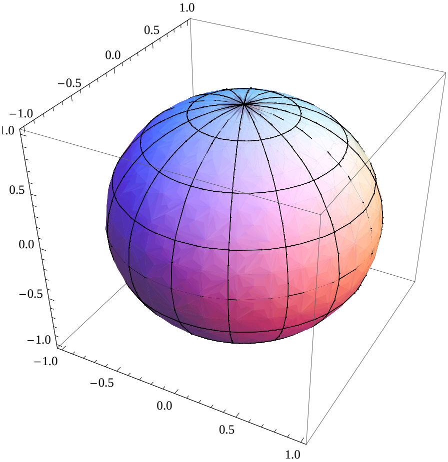
Cone: $x^2 + z^2 = y^2$, $0 \leq y \leq 1$

Simple 3D Surfaces



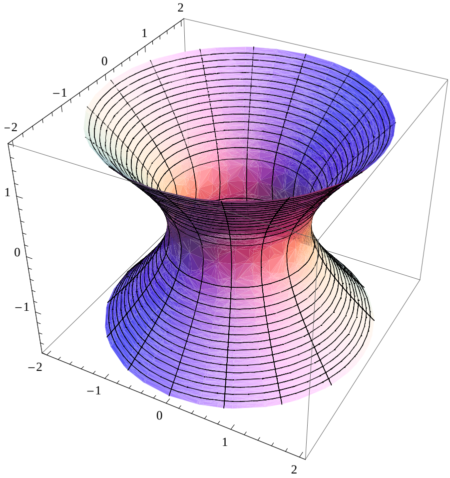
Paraboloid: $x^2 + z^2 = y$, $0 \leq y \leq 1$

Simple 3D Surfaces



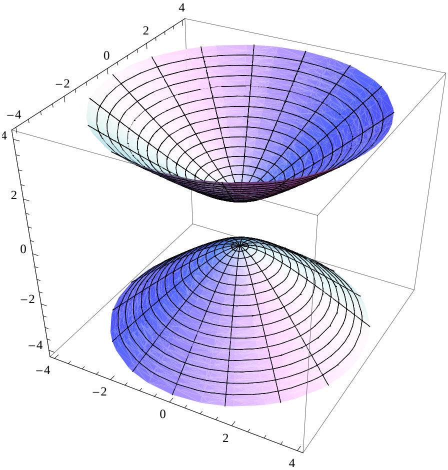
Sphere: $x^2 + y^2 + z^2 = 1$

Simple 3D Surfaces



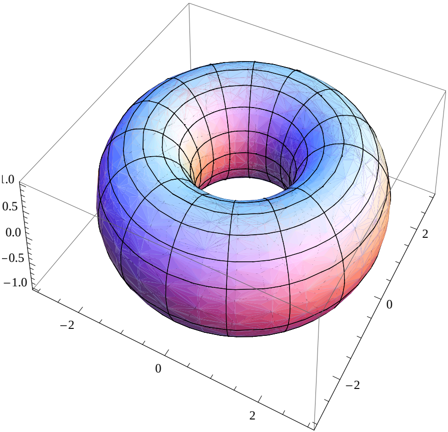
Hyperboloid of one sheet: $x^2 + z^2 = y^2 + 1$, $-\sqrt{3} \leq y \leq \sqrt{3}$

Simple 3D Surfaces



Hyperboloid of two sheets: $x^2 + z^2 = y^2 - 1$, $-\sqrt{17} \leq y \leq \sqrt{17}$

Simple 3D Surfaces



Torus: $4(x^2 + z^2) = \left[(r^2 - 1) - (x^2 + y^2 + z^2)\right]^2$

Outline

Surfaces Defined by Functions

- Let $y = f(x, z)$ be a function of two variables.
- The function $f(x, z)$ gives the height of the surface over the point $(x, 0, z)$ in the xz-plane.
- For every point (x, z) in the plane $y = 0$, the function produces a y-coordinate, giving a point (x, y, z) in space.

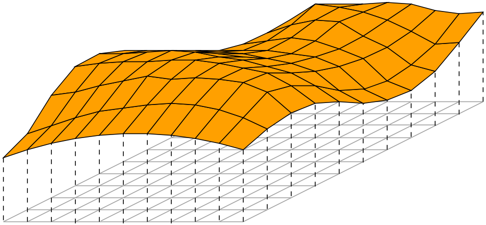
Examples

Examples (Surfaces Defined by Functions)

- The function $f(x, z) = 4 - x - z$ defines a plane.
- The function $f(x, z) = \sqrt{1 - x^2 - z^2}$ defines a hemisphere.
- The function $f(x, z) = \sqrt{1 - x^2}$ defines a half-cylinder.

Creating a Rectangular Mesh

- We can create a rectangular mesh for a surface represented by $y = f(x, z)$ over a rectangular region $[a, b] \times [c, d]$.
- Subdivide the range $[a, b]$ of values of x into $a = x_0, x_1, \ldots, x_m = b$.
- Subdivide the range $[c, d]$ of values of z into $c = z_0, z_1, \ldots, z_n = d$.



Outline

Normals

- In order for the lighting effects to be computed properly, we must also create a unit normal vector at each grid point.
- This normal should be perpendicular to the tangent plane.
- To compute it, we take the cross product of two vectors lying in the tangent plane.

Tangent Planes

- Let **u** be the tangent vector parallel to the *yz*-plane, and therefore perpendicular to the *x*-axis.
  - It is constant in the *x*-direction.
  - Its "slope," or rate of change, of y in the z direction, is $\frac{\partial f}{\partial z}$.
- Therefore,

$$\mathbf{u} = \left(0, \frac{\partial f}{\partial z}, 1\right).$$

- Similarly, if **v** is the tangent vector parallel to the *xy*-plane, then

$$\mathbf{v} = \left(1, \frac{\partial f}{\partial x}, 0\right).$$

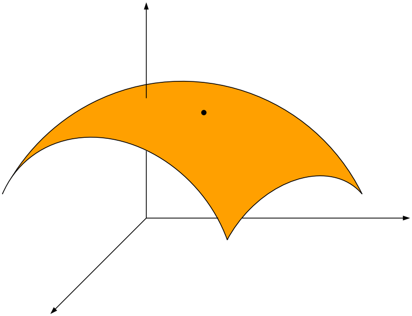
The Normal Vector

- Thus, a normal vector $\mathbf{n}$ is given by

$$\begin{aligned}\mathbf{n} &= \mathbf{u} \times \mathbf{v} \\ &= \left(0, \frac{\partial f}{\partial z}, 1\right) \times \left(1, \frac{\partial f}{\partial x}, 0\right) \\ &= \left(-\frac{\partial f}{\partial x}, 1, -\frac{\partial f}{\partial z}\right).\end{aligned}$$

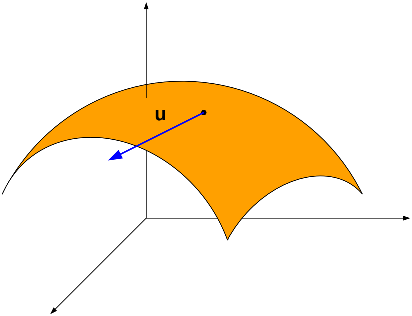
- Normalize this to the unit vector $\mathbf{N} = \mathit{normalize}(\mathbf{n})$.

The Normal Vector



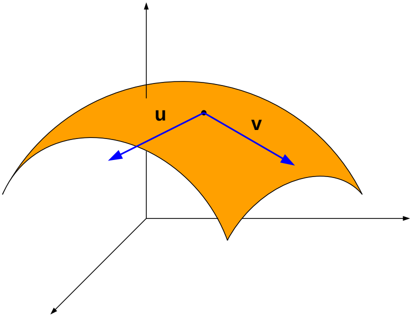
The surface

The Normal Vector



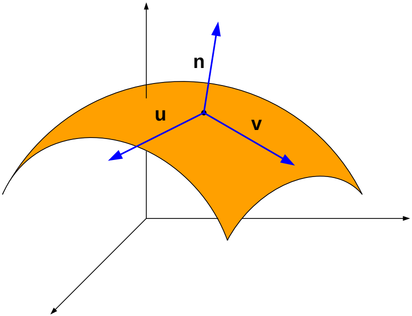
The vector $\mathbf{u} = \left(0, \frac{\partial f}{\partial z}, 1\right)$

The Normal Vector



The vector $\mathbf{v} = \left(1, \frac{\partial f}{\partial x}, 0\right)$

The Normal Vector



The normal vector $\mathbf{n} = \left(-\frac{\partial f}{\partial x}, 1, -\frac{\partial f}{\partial z}\right)$

Outline

Review of Derivatives

Function	Derivative
x^n	nx^{n-1}
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$f(x)^n$	$nf(x)^n f'(x)$
$\sqrt{f(x)}$	$\frac{f'(x)}{2\sqrt{f(x)}}$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$

A Hemisphere

Example (A Hemisphere)

- Let $y = \sqrt{1 - x^2 - z^2}$.
- Then
$$\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{1 - x^2 - z^2}} = -\frac{x}{y}$$
and
$$\frac{\partial f}{\partial z} = -\frac{z}{\sqrt{1 - x^2 - z^2}} = -\frac{z}{y}.$$
- Therefore,
$$\mathbf{n} = \left(\frac{x}{y}, 1, \frac{z}{y}\right).$$

Example (A Hemisphere)

- Then
$$|\mathbf{n}| = \sqrt{\frac{x^2}{y^2} + 1 + \frac{z^2}{y^2}} = \sqrt{\frac{x^2 + y^2 + z^2}{y^2}} = \frac{1}{y}.$$
- The normalized vector is $\mathbf{N} = (x, y, z)$.

Example (The Paraboloid)

- Let $y = x^2 + z^2$.
- Then
$$\frac{\partial f}{\partial x} = 2x$$
and
$$\frac{\partial f}{\partial z} = 2z.$$
- Therefore,
$$\mathbf{n} = (-2x, 1, -2z).$$

The Paraboloid

Example (The Paraboloid)

- Then
$$|\mathbf{n}| = \sqrt{4x^2 + 1 + 4z^2} = \sqrt{1 + 4(x^2 + z^2)} = \sqrt{1 + 4y}.$$
- The normalized vector is
$$\mathbf{N} = \left(-\frac{2x}{\sqrt{1+4y}}, \frac{1}{\sqrt{1+4y}}, -\frac{2z}{\sqrt{1+4y}}\right).$$

Outline

Assignment

Assignment

- Assignment 17.